数理解析研究所講究録
1531巻 2007年 1-4

ON A PROBLEM OF GUTEV, OHTA AND YAMAZAKI CONCERNING CONTINUOUS SELECTIONS

島根大学総合理工学部 山内 貴光 (Takamitsu Yamauchi)
Department of Mathematics and Computer Sciences
Shimane University

Throughout this note, all spaces are assumed to be T_1. For undefined terminology, we refer to [2]. The purpose of this note is to introduce some results of [9] and [10].

Let X be a space and $(Y, \|\cdot\|)$ a Banach space. By 2^Y, $\mathcal{F}_c(Y)$, $\mathcal{C}_c(Y)$ and $\mathcal{C}'_c(Y)$ we denote the set of all non-empty subsets of Y, the set of all non-empty closed convex subsets of Y, the set of all non-empty compact convex subsets of Y and the set $\mathcal{C}_c(Y) \cup \{Y\}$, respectively. Then a mapping $\varphi : X \to 2^Y$, which is called a set-valued mapping from X to Y, associates each point $x \in X$ with a non-empty subset $\varphi(x)$ of Y. For a mapping $\varphi : X \to 2^Y$, a mapping $f : X \to Y$ is called a *selection* if $f(x) \in \varphi(x)$ for each $x \in X$.

For $K \in \mathcal{F}_c(Y)$, a point $y \in K$ is called an *extreme point* if every open line segment containing y is not contained in K. For $K \in \mathcal{F}_c(Y)$, the *weak convex interior* $\mathrm{wci}(K)$ of K ([3]) is the set of all non-extreme points of K, that is,

$$\mathrm{wci}(K) = \{y \in K \mid y = \delta y_1 + (1-\delta)y_2 \text{ for some } y_1, y_2 \in K \setminus \{y\} \text{ and } 0 < \delta < 1\}.$$

Our concern of this note is to characterize some topological properties in terms of continuous selections avoiding extreme points. This study is motivated by Problem 3 below posed by V. Gutev, H. Ohta and K. Yamazaki [3].

1 A problem of Gutev, Ohta and Yamazaki

By $w(Y)$ we denote the weight of a space Y. A Hausdorff space X is called *countably paracompact* if every countable open cover of X is refined by a locally finite open cover of X. The following insertion theorem due to C. H. Dowker [1, Theorem 4] and M. Katětov [4, Theorem 2] is fundamental.

Theorem 1 (Dowker [1], Katětov [4]). *A T_1-space X is normal and countably paracompact if and only if for every upper semicontinuous function $g : X \to \mathbf{R}$ and every lower semicontinuous function $h : X \to \mathbf{R}$ with $g(x) < h(x)$ for each $x \in X$, there exists a continuous function $f : X \to \mathbf{R}$ such that $g(x) < f(x) < h(x)$ for each $x \in X$.*

The cardinality of a set S is denoted by $\mathrm{Card}\, S$. For an infinite cardinal number λ, a T_1-space X is called *λ-collectionwise normal* if for every discrete collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of X with $\mathrm{Card}\, A \leq \lambda$, there exists a disjoint collection $\{G_\alpha \mid \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$. A mapping $\varphi : X \to 2^Y$ is called *lower semicontinuous* (*l.s.c.* for short) if for every open subset V of Y, the set $\varphi^{-1}[V] = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$ is open in X. Let $\mathbf{R}$ be the space of real numbers with the usual topology. The space $c_0(\lambda)$ is the Banach space consisting of functions $s : D(\lambda) \to \mathbf{R}$, where $D(\lambda)$ is a set with $\operatorname{Card} D(\lambda) = \lambda$, such that for each $\varepsilon > 0$ the set $\{\alpha \in D(\lambda) \mid |s(\alpha)| \geq \varepsilon\}$ is finite, where the linear operations are defined pointwise and $\|s\| = \sup\{|s(\alpha)| \mid \alpha \in D(\lambda)\}$ for each $s \in c_0(\lambda)$. In order to connect insertion theorems with selection theorems, V. Gutev, H. Ohta and K. Yamazaki [3] introduced lower and upper semicontinuity of a mapping to the Banach space $c_0(\lambda)$ and, with the aid of these concepts, they proved sandwich-like characterizations of paracompact-like properties. Moreover, they introduced generalized $c_0(\lambda)$-spaces for Banach spaces and established the following theorem [3, Theorem 4.5].

Theorem 2 (Gutev, Ohta and Yamazaki [3]). *For a T_1-space X, the following statements are equivalent.*

(*a*) *X is countably paracompact and λ-collectionwise normal.*

(*b*) *For every generalized $c_0(\lambda)$-space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with $\operatorname{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

(*c*) *For every closed subset A of X and every two mappings $g, h : A \to c_0(\lambda)$ such that g is upper semicontinuous, h is lower semicontinuous and $g(x) < h(x)$ for each $x \in A$, there exists a continuous mapping $f : X \to c_0(\lambda)$ such that $g(x) < f(x) < h(x)$ for each $x \in A$.*

Concerning this theorem, they posed the following problem [3, Problem 4.7]:

Problem 3 (Gutev, Ohta and Yamazaki [3]). *Can "every generalized $c_0(\lambda)$-space Y" in condition (b) of Theorem 2 be replaced by "every Banach space Y with $w(Y) \leq \lambda$"?*

It is proved in [9] that the answer of Problem 3 is affirmative.

Theorem 4 ([9]). *A T_1-space X is countably paracompact and λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with $\operatorname{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

In particular, we have the following.

Corollary 5. *A T_1-space X is countably paracompact and collectionwise normal if and only if for every Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$ with $\operatorname{Card}\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \operatorname{wci}(\varphi(x))$ for each $x \in X$.*

Comparing Corollary 5 with selection theorems due to E. Michael [6] and S. Nedev [7], it is natural to ask whether other topological properties such as paracompactness can be characterized analogously. In the next section, we present some characterizations in terms of continuous selections avoiding extreme points.

2 Characterizations in terms of continuous selections avoiding extreme points

For an infinite cardinal number λ, a Hausdorff space X is called *λ-paracompact* if every open cover $\mathcal{U}$ of X with $\mathrm{Card}\,\mathcal{U} \leq \lambda$ is refined by a locally finite open cover of X. The following theorem is a λ-paracompact analogue of Theorems 2 and 4.

Theorem 6 ([9]). *A T_1-space X is normal and λ-paracompact if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Thus we have the following variation of [6, Theorem 3.2″].

Corollary 7. *A T_1-space X is paracompact if and only if for every Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ such that $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

For an infinite cardinal number λ, a space X is *λ-PF-normal* if every point-finite open cover $\mathcal{U}$ of X with $\mathrm{Card}\,\mathcal{U} \leq \lambda$ is normal. A space X is called *PF-normal* if X is λ-PF-normal for every infinite cardinal λ. Every λ-collectionwise normal space is λ-PF-normal, and ω-PF-normality coincides with normality ([5, Theorem 2], [8, Theorem 3.2]). Note that PF-normality is not hereditary to closed subsets ([3, p.506], [8, p. 409]), but it is hereditary to open F_σ-subsets.

Theorem 8 ([10]). *A T_1-space X is countably paracompact and λ-PF-normal if and only if for every Banach space Y with $w(Y) \leq \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Corollary 9. *A T_1-space X is countably paracompact and PF-normal if and only if for every Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Theorems 6 and 8 provide the following variation of [6, Theorem 3.1″].

Corollary 10. *For a T_1-space X, the following statements are equivalent.*

(a) *X is normal and countably paracompact.*

(b) *For every separable Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

(c) *For every separable Banach space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$ with $\mathrm{Card}\,\varphi(x) > 1$ for each $x \in X$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$.*

Applying Theorem 2, V. Gutev, H. Ohta and K. Yamazaki [3, Theorem 4.6] proved that a T_1-space X is perfectly normal and λ-collectionwise normal if and only if for every generalized $c_0(\lambda)$-space Y and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$ with $\mathrm{Card}\,\varphi(x) > 1$. By applying Theorem 4, instead of Theorem 2, to the proof of [3, Theorem 4.6], we have the following corollary.

Corollary 11. *A T_1-space X is perfectly normal and λ-collectionwise normal if and only if for every Banach space Y with $w(Y) \le \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{C}'_c(Y)$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$ with $\mathrm{Card}\,\varphi(x) > 1$.*

Analogously, we have the following.

Corollary 12. *A T_1-space X is perfectly normal and λ-paracompact if and only if for every Banach space Y with $w(Y) \le \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{F}_c(Y)$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$ with $\mathrm{Card}\,\varphi(x) > 1$.*

Corollary 13. *A T_1-space X is perfectly normal and λ-PF-normal if and only if for every Banach space Y with $w(Y) \le \lambda$ and every l.s.c. mapping $\varphi : X \to \mathcal{C}_c(Y)$, there exists a continuous selection $f : X \to Y$ of φ such that $f(x) \in \mathrm{wci}(\varphi(x))$ for each $x \in X$ with $\mathrm{Card}\,\varphi(x) > 1$.*

References

[1] C. H. Dowker, *On countably paracompact spaces*, Canad. J. Math. **3** (1951), 219–224.

[2] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[3] V. Gutev, H. Ohta and K. Yamazaki, *Selections and sandwich-like properties via semi-continuous Banach-valued functions*, J. Math. Soc. Japan **55** (2003), 499–521.

[4] M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. **38** (1951), 85–91.

[5] E. Michael, *Point-finite and locally finite coverings*, Canad. J. Math. **7** (1955), 275–279.

[6] E. Michael, *Continuous selections* I, Ann. of Math. **63** (1956), 361–382.

[7] S. Nedev, *Selection and factorization theorems for set-valued mappings*, Serdica **6** (1980), 291–317.

[8] J. C. Smith, *Properties of expandable spaces*, General topology and its relations to modern analysis and algebra, III (Proc. Third Prague Topological Sympos., 1971), Academia, Prague, 1972, 405–410

[9] T. Yamauchi, *Continuous selections avoiding extreme points*, Topology Appl. (to appear).

[10] T. Yamauchi, *Seleciton theorems on spaces in which every point-finite open cover is normal*, preprint.

Department of Mathematics, Shimane University, Matsue, 690-8504, Japan
E-mail address: `t_yamauchi@riko.shimane-u.ac.jp`